Optimal Linear Smoothing

Data collected over $[0,T]$. Find best estimate $\hat{x}(t) \mid y_T$.

Form:

$$\hat{\underline{x}}(t|T) = \underline{A}\,\hat{\underline{x}}(t) + \underline{A}'\,\hat{\underline{x}}_b(t)$$

where $\underline{A}, \underline{A}'$ are weighting matrices

$\hat{\underline{x}}(t)$ is result of forward filter using y_t.

$\hat{\underline{x}}_b(t)$ is result of backward filter using $y_\tau;\ t < \tau \le T$.

$$\underline{e}(t|T) = \hat{\underline{x}}(t|T) - \underline{x}(t)$$

$$= \underbrace{\underline{A}\,\hat{\underline{x}}(t)}_{x + e_f(t)} + \underline{A}'\underbrace{\hat{x}_b(t)}_{x + e_b(t)} - \underline{I}\,\underline{x}(t)$$

$$= [\underline{A} + \underline{A}' - \underline{I}]\,\underline{x}(t) + \underline{A}\,\underline{e}_f(t) + \underline{A}'e_b(t)$$

everything unbiased $\Rightarrow$ $\underline{A} + \underline{A}' - \underline{I} = \underline{0}$

or $\underline{A}' = \underline{I} - \underline{A}$

$$\hat{\underline{x}}(t|T) = \underline{A}\,\hat{\underline{x}}(t) + (\underline{I} - \underline{A})\,\hat{\underline{x}}_b(t)$$

$$\underline{\Sigma}(t|T) = E\{(\underline{A}\,\underline{e}_f(t) + (\underline{I}-\underline{A})\underline{e}_b(t))(\underline{A}\,\underline{e}_f(t) + (\underline{I}-\underline{A})\underline{e}_b(t))^T\}$$

$$= \underline{A}\,\underline{\Sigma}(t)\,\underline{A}^T + (\underline{I}-\underline{A})\,\underline{\Sigma}_b(t)(\underline{I}-\underline{A})^T$$

to minimize $tr[\underline{\Sigma}(t|T)]$: $\frac{\partial \Sigma(t|T)}{\partial \underline{A}} = 0$:

$$\underline{0} = 2\underline{A}\,\underline{\Sigma}(t) + 2(\underline{I}-\underline{A})\underline{\Sigma}_b(t)(-\underline{I})$$

or $$\underline{A} = \underline{\Sigma}_b(t)[\underline{\Sigma}(t) + \underline{\Sigma}_b(t)]^{-1}$$

and
$$\underline{I}-\underline{A} = [\underline{\Sigma}(t) + \underline{\Sigma}_b(t)]^{-1}\underline{\Sigma}(t)$$
$$= \underline{\Sigma}(t)[\underline{\Sigma}(t) + \underline{\Sigma}_b(t)]^{-1}$$

substituting back into orig Σ eqn & simplifying:

$$\underline{\Sigma}^{-1}(t|T) = \underline{\Sigma}^{-1}(t) + \underline{\Sigma}_b^{-1}(t)$$

$\Rightarrow$ $\underline{\Sigma}(t|T) \leq \underline{\Sigma}(t)$, or estimate is at least as good as KBF result.

$$\hat{\underline{x}}(t|T) = \underline{A}\,\hat{\underline{x}}(t) + (\underline{I}-\underline{A})\hat{\underline{x}}_b(t)$$

$$= \underline{\Sigma}(t|T)[\underline{\Sigma}^{-1}(t)\hat{\underline{x}}(t) + \underline{\Sigma}_b^{-1}(t)\hat{\underline{x}}_b(t)]$$

— weighted sum of the two estimates.

$$2P_b (P+P_b)^{-1} P + 2(\underline{I} - \underline{A}) \underline{P}_b(-\underline{I}) = 0$$

$$\underline{I} - \underline{A} = \underline{P}_b \underline{P}_b^{-1} (\underline{P} + \underline{P}_b)^{-1} \underline{P} = (\underline{P} - \underline{P}_b)^{-1} \underline{P}$$

Linear system:

$$\dot{\underline{x}} = \underline{F}\underline{x} + \underline{G}\underline{w} \qquad \underline{w} \sim N(\underline{0}, \underline{Q})$$

$$\underline{z} = \underline{H}\underline{x} + \underline{v} \qquad \underline{v} \sim N(\underline{0}, \underline{R})$$

KBF is forward filter:

$$\dot{\hat{\underline{x}}}(t) = \underline{F}\hat{\underline{x}}(t) + \underline{\Sigma}(t)\underline{H}^T\underline{R}^{-1}[\underline{z} - \underline{H}\hat{\underline{x}}] \qquad \hat{\underline{x}}(0) = E\{\underline{x}(0)\}$$

$$= (\underline{F} - \underline{K}\underline{H})\hat{\underline{x}}(t) + \underline{K}(t)\underline{z}(t)$$

$$\dot{\underline{\Sigma}}(t) = \underline{F}\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{F}^T + \underline{G}\underline{Q}\underline{G}^T - \underline{\Sigma}(t)\underline{H}^T\underline{R}^{-1}\underline{H}\underline{\Sigma}(t)$$

$$\underline{\Sigma}(0) = \underline{\Sigma}_0$$

Backward filter: set $\tau = T - t$
& apply KBF eqns.

$$\frac{d}{d\tau}\hat{\underline{x}}_b = -\underline{F}\hat{\underline{x}}_b + \underline{\Sigma}_b(t)\underline{H}^T\underline{R}^{-1}[\underline{z} - \underline{H}\hat{\underline{x}}_b]$$

$$\frac{d}{d\tau}\underline{\Sigma}_b = -\underline{F}\underline{\Sigma}_b - \underline{\Sigma}_b\underline{F}^T + \underline{G}\underline{Q}\underline{G}^T - \underline{\Sigma}_b\underline{H}^T\underline{R}^{-1}\underline{H}\underline{\Sigma}_b$$

but, $\hat{\underline{x}}_b(T)$ completely unknown $\Rightarrow$ $\underline{\Sigma}_b^{-1}(T) = 0$
& can't use this last eqn numerically.

Set

$$\underline{s}(t) \triangleq \underline{\Sigma}_b^{-1}(t)\,\underline{\hat{x}}_b(t)$$

since $\underline{\hat{x}}_b(T)$ finite, $\underline{s}(T) = \underline{0}$

$$\frac{d}{d\tau}\underline{\Sigma}_b^{-1} = -\underline{\Sigma}_b^{-1}\left(\frac{d}{d\tau}\underline{\Sigma}_b\right)\underline{\Sigma}_b^{-1}$$

$\Downarrow$

$$\frac{d}{d\tau}\underline{\Sigma}_b^{-1} = \underline{\Sigma}_b^{-1}\underline{F} + \underline{F}^T\underline{\Sigma}_b^{-1} - \underline{\Sigma}_b^{-1}\underline{G}\,\underline{Q}\,\underline{G}^T\underline{\Sigma}_b^{-1} + \underline{H}^T\underline{R}^{-1}\underline{H}$$

and

$$\frac{d}{d\tau}\underline{s} = \left(\underline{F}^T - \underline{\Sigma}_b^{-1}\underline{G}\,\underline{Q}\,\underline{G}^T\right)\underline{s}(t) + \underline{H}^T\underline{R}^{-1}\underline{z}(t)$$

Table 1 Summary of Smoothers.

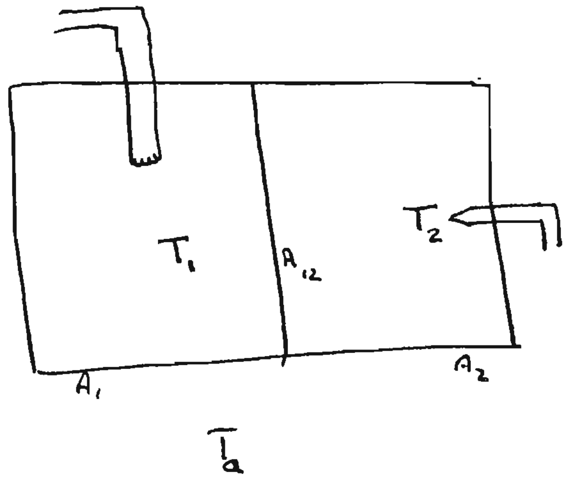
Smoothability: Smoothable if $\underline{\Sigma}(t|T) < \underline{\Sigma}(t)$.

Smoothable $\Rightarrow$ $(F, G\sqrt{Q})$ controllable.

Three types of smoothers:

1) Fixed interval
2) Fixed point
3) Fixed lag.

EXAMPLE



$$\dot{T}_1 = -c(T_1 - T_2)A_{12} - c(T_1 - T_a)A_1 + bu(t)$$

$$\dot{T}_2 = -c(T_2 - T_1)A_{12} - c(T_2 - T_a)A_2$$

$$y = T_2$$

state is $\begin{bmatrix} T_1 \\ T_2 \\ T_a \end{bmatrix}$

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_a \end{bmatrix} = \begin{bmatrix} -c[A_{12}+A_1] & cA_{12} & cA_1 \\ cA_{12} & -c[A_{12}+A_2] & cA_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$+ \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} (u(t) + w_1(t)) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_2(t)$$

$$= \underline{A}\,\underline{T} + \underline{B}\,\underline{u}(t) + \begin{bmatrix} bw_1 \\ 0 \\ w_2 \end{bmatrix}$$

$$E\left\{ \begin{bmatrix} bw_1 \\ 0 \\ w_2 \end{bmatrix} (bw_1 \;\; 0 \;\; w_2) \right\} = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_{33} \end{bmatrix}$$

$$= \underline{Q}$$

$$y = T_2 + v(t) \qquad E\{v(t)v^T(t)\} = \underline{R}$$