Birdwell

Linear System
Part I

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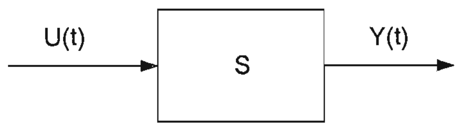
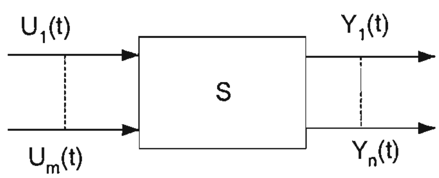
Outline

- State-space notation
- Transition matrix

State Space Notation



SISO system



MIMO system

State space model provides a method to study the dynamic system. Moreover, this approach to system description is closer to physical reality than any of the frequency-oriented transform techniques.

Linear State Equation

The basic representation for linear systems is the linear state equation, customarily written in the standard form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

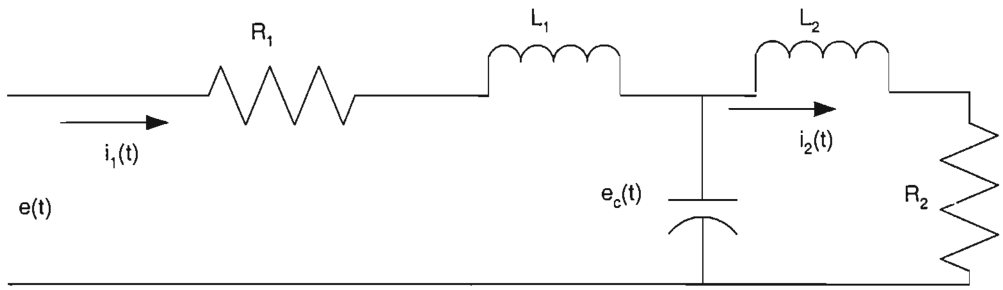
$x(t)$ is called the state vector, and its components, $x_1(t),\ldots x_n(t)$, are the state variables.

$u(t)$ is the input signal

$y(t)$ is the output signal

$A(t), B(t), C(t), D(t)$ are matrices

Example 1



Electric Circuit System

According to the Kirchhoff's law, the electric circuit can be presented by:

$$e(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + e_c(t)$$

$$e_c(t) = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt}$$

$$C \frac{de_c(t)}{dt} = i_1(t) - i_2(t)$$

Example 1

If use $x_1(t) = i_1(t), x_2(t) = i_2(t), x_3(t) = e_c(t)$ as state variables and $u(t) = e(t), y(t) = e_c(t)$ state space model is represented by:

$$\dot{x}_1(t) = -\frac{R_1}{L_1} x_1(t) - \frac{1}{L_1} x_3(t) + \frac{1}{L_1} u(t)$$

$$\dot{x}_2(t) = -\frac{R_2}{L_2} x_2(t) + \frac{1}{L_2} x_3(t)$$

$$\dot{x}_3(t) = \frac{1}{C} x_1(t) - \frac{1}{C} x_2(t)$$

The output equation is

$$y(t) = x_3(t)$$

Example 1

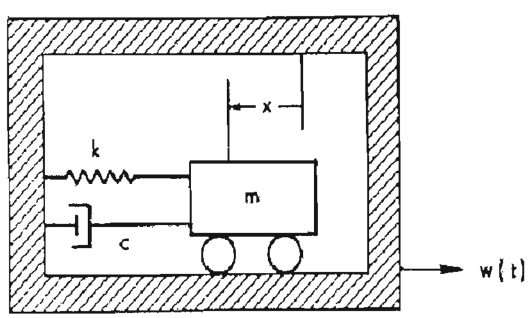
Let $x(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$

Then state-space equation is
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

$$A = \begin{bmatrix} \frac{-R_1}{L_1} & 0 & \frac{-1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{c} & \frac{-1}{c} & 0 \end{bmatrix} \qquad B = \frac{1}{L_1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = [0 \quad 0 \quad 1]$$

Example 2

Consider the mass m shown below. It is connected to the left wall by a spring with spring constant k and a damper with damping coefficient c. Frictionless wheels are assumed. Displacement x is measured (positive-left) between the indications; the entire container is subject to acceleration w(t) which is positive to the right. This is a one-dimensional translation-motion-only system and, consequently, displacement x and velocity dx/dt are suitable state variables.



Example 2

The equation of motion of the system is obtained from Newton's second law: $\sum f_x = ma$

The forces acting are $\sum f_x = -kx - c\dot{x}$ Corresponding to the spring and damper. Total acceleration is $a = \ddot{x} - w(t)$ so that

$$m\ddot{x} + c\dot{x} + kx = mw(t)$$

If the state vector is defined as $\bar{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

The appropriate equation for the system dynamics is

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \end{bmatrix}$$

Example 3

Given an n^{th}-order linear differential equation:

$$[D^n + a_{n-1}(t)D^{n-1} + \ldots a_1(t)D + a_0(t)]y(t) = w(t) \quad (1\text{-}1)$$

Where D=d/dt, we can define a set of state variables $x_1(t),\ldots x_n(t)$ by

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \dot{x}_1(t) \\ &\vdots \\ x_n(t) &= x_{n-1}(t) \end{aligned}$$

Example 3

These relations can be written as a set of n first-order linear differential equations:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
\dot{x}_n(t) &= -a_0(t)x_1(t) - a_1(t)x_2(t) - \cdots - a_{n-1}(t)x_n(t) + w(t)
\end{aligned}$$

The first n-1 of these equations follow from the state variable definitions;

The last one is obtained using the definitions and equation (1-1)

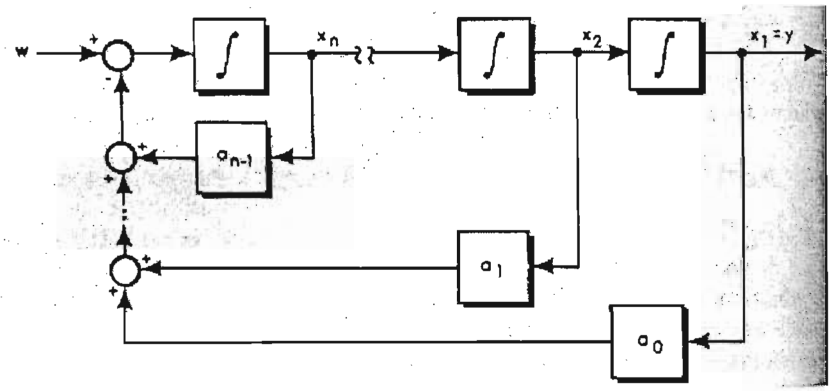
Example 3

Expressing the equation in vector-matrix form yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w \end{bmatrix} \quad (1\text{-}2)$$

This is called the companion form of (1-1)

Example 3



Block diagram representation of equation (1-2)

Transition Matrix

The basic questions of existence and uniqueness of solutions are first addressed for linear state equations unencumbered by inputs and outputs. That is

$$\dot{x}(t) = A(t)x(t), x(t_0) = x_0 \qquad (2\text{-}1)$$

$$x(t) = \left[I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \cdots + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \cdots \int_{t_0}^{\sigma_{k-1}} A(\sigma_k)d\sigma_k \cdots d\sigma_1 \cdots \right] x_0$$

Denoting the n-by-n matrix series on the right side by $\Phi(t,t_0)$

The solution just constructed can be written in terms of this transition matrix as $x(t) = \Phi(t,t_0)x_0$

Transition Matrix

It is convenient for some purposes to view the transition matrix as a function of two variables, written as $\Phi(t,\tau)$

Defined by the Peano-Baker series:

$$\Phi(t,\tau) = I + \int_{\tau}^{t} A(\sigma_1)d\sigma_1 + \int_{\tau}^{t} A(\sigma_1)\int_{\tau}^{t} A(\sigma_2)d\sigma_2 d\sigma_1 \cdots$$

Transition Matrix Properties

Property 1: If A(t)=A, an n×n constant matrix, then the transition matrix is

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

Where the matrix exponential is defined by the power series:

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

That converges uniformly and absolutely on [-T,T], where T>0 is arbitrary

Transition Matrix Properties

Property 2:

If for every t and τ ,

$$A(t)\int_{\tau}^{t} A(\sigma)d\sigma = \int_{\tau}^{t} A(\sigma)d\sigma A(t)$$

then

$$\Phi(t,\tau) = e^{\int_{\tau}^{t} A(\sigma)d\sigma} = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\int_{\tau}^{t} A(\sigma)d\sigma\right]^k$$

Transition Matrix Properties

Property 3: The linear $n \times n$ matrix differential equation

$$\frac{d}{dt}X(t) = A(t)X(t), X(t_0) = I$$

has the unique, continuously-differentiable solution

$$X(t) = \Phi_A(t,t_0)$$

Transition Matrix Properties

Property 4: The linear n × n matrix differential equation

$$\frac{d}{dt}Z(t) = -A^T(t)Z(t), Z(t_0) = I$$

has the unique, continuously-differentiable solution

$$Z(t) = \Phi_A^T(t_0, t)$$

Transition Matrix Properties

Property 5: For every t, τ , σ ,the transition matrix for A(t) satisfies

$$\Phi(t,\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau)$$

Example: let $\tau = t_0, \sigma = t_1 > t_0, t = t_2 > t_1$. then the composition

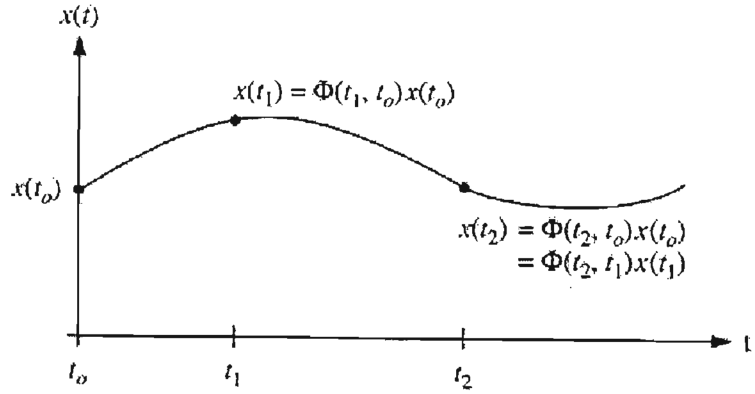
property implies that the solution of (2-1) at time t_2 can be represented as

$$x(t_2) = \Phi(t_2, t_0)x(t_0)$$

or as $\quad x(t_2) = \Phi(t_2, t_1)x(t_1)$

where $\quad x(t_1) = \Phi(t_1, t_0)x(t_0)$

An Illustration of the Composition Property



Transition Matrix Properties

Property 6: For every t and τ the transition matrix for A(t) satisfies

$$\det \Phi(t,\tau) = e^{\int_{\tau}^{t} tr[A(\sigma)]d\sigma}$$

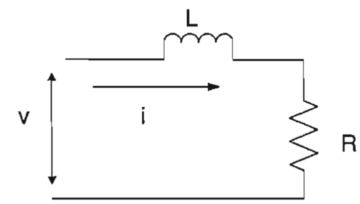
Property 7: The transition matrix for A(t) is invertible for every t and τ , and

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t)$$

Example 4

Consider the circuit shown below, which is composed of a voltage source, v, a resistor, R, and an inductance,L. Kirchhoff's voltage law yields

$$v = iR + L\frac{di}{dt}$$



Elementary Electrical Circuit

Example 4

We assume $i=i_0$ at $t=t_0$ and $v=0$ for all time, which yields:

$$\frac{di}{dt} = -\frac{R}{L}i$$

The system dynamics matrix A is merely the scalar quantity –R/L. Elementary differential equation solution techniques yield:

$$i(t) = i_0 e^{-\frac{R}{L}(t-t_0)}$$

Reference

- Wilson J. Rugh, “Linear System Theory”, second edition
- Chi-Tsong Chen, “Linear System Theory and Design”, third edition
- Arthur gelb, “Applied Optimal Estimation”.

Example 4

From the solution we identify the transition matrix as:

$$\Phi(t,t_0) = e^{-\frac{R}{L}(t-t_0)}$$

Property 5 of the transition matrix are readily verified.
For times t_0, t_1, t_2, we write

$$\Phi(t_2,t_1) = e^{-\frac{R}{L}(t_2-t_1)}$$

$$\Phi(t_1,t_0) = e^{-\frac{R}{L}(t_1-t_0)}$$

Example 4

So that:

$$\Phi(t_2,t_1)\Phi(t_1,t_0) = e^{-\frac{R}{L}(t_2-t_1)} e^{-\frac{R}{L}(t_1-t_0)}$$

$$= e^{-\frac{R}{L}(t_2-t_0)}$$

$$= \Phi(t_2,t_0)$$

LINEAR DYNAMIC SYSTEMS (II)

Kaiyu Wang

1 MATRIX SUPERPOSITION INTEGRAL

Consider first the linear system including forcing function inputs

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + L(t)\underline{u}(t) \tag{1}$$

The complete solution for this system is

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)\underline{u}(\tau)d\tau \tag{2}$$

Proof. It can be concluded from linear system theory that the transition matrix Φ is invertible. Let

$$\underline{z}(t) = P^{-1}(t)\underline{x}(t) = \Phi^{-1}(t, t_o)\underline{x}(t), \quad \underline{x}(t_0) = \underline{x}_0$$

Substituting into (1) yields

$$F(t)\Phi(t, t_0)\underline{z}(t) + \Phi(t, t_0)\dot{\underline{z}}(t) = F(t)\Phi(t, t_0)\underline{z}(t) + L(t)\underline{u}(t), \qquad \underline{z}(t_0) = \underline{x}_0$$

or

$$\dot{\underline{z}}(t) = \Phi^{-1}(t, t_0)L(t)\underline{u}(t), \quad \underline{z}(t_0) = \underline{x}_0$$

Both sides can be integrated from t_0 to t to obtain

$$\underline{z}(t) - \underline{x}_0 = \int_{t_0}^{t} \Phi^{-1}(\tau, t_0)L(\tau)\underline{u}(\tau)d\tau$$

Replacing $\underline{z}(t)$ by $\Phi^{-1}(t, t_o)\underline{x}(t)$ and rearranging using properties of the transition matrix gives

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)\underline{u}(\tau)d\tau \tag{3}$$

Equation (3) is often called the matrix superposition integral. ■

The solution of the dynamics equation, when a random input is present, proceeds in an analogous fashion. Thus, corresponding to

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + G(t)\underline{w}(t) + L(t)\underline{u}(t)$$

we directly find

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)\underline{w}(\tau)d\tau + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)\underline{u}(\tau)d\tau \tag{4}$$

EXAMPLE

The electrical circuit of Fig.1 is composed of a voltage source, v, a resistor, R and an inductor, L. Kirchhoff's voltage law yields

$$v = iR + L\frac{di}{dt} \tag{5}$$

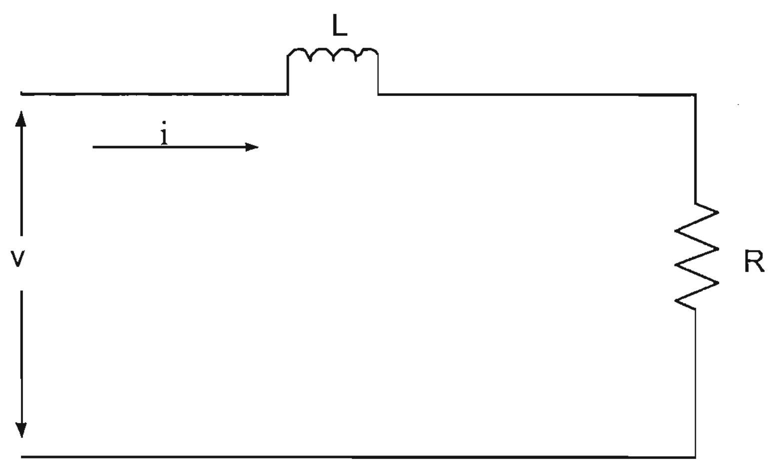


Fig.1 Elementary Electrical Circuit

we assume $i = i_0$ at $t = t_0$, if $v = 0$ for all time, the system will be

$$\frac{di}{dt} = -\frac{R}{L}i$$

The transition matrix can be verified which is

$$\Phi(t, t_0) = e^{-\frac{R}{L}(t-t_0)}$$

or

$$\Phi(t, 0) = e^{-\frac{R}{L}t}$$

From Eq. 2, the solution of Eq. 5, considering the input voltage, will be

$$\begin{aligned} i(t) &= i_0 e^{-\frac{R}{L}t} + \int_0^t e^{-\frac{R}{L}(t-\tau)} \frac{v}{L} d\tau \\ &= i_0 e^{-\frac{R}{L}t} + \frac{v}{R}(1 - e^{-\frac{R}{L}t}) \end{aligned}$$

2 DISCRETE FORMULATION

Consider the continuous-time state equation

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + G(t)\underline{w}(t) + L(t)\underline{u}(t)$$

If the set of equations is to be completed on a digital computer, it must be discretized. At discrete points in time, t_k, $k = 1, 2, ...$, the resulting difference equation is, from Eq. (4),

$$\underline{x}_{k+1} = \Phi_k \underline{x}_k + \Gamma_k \underline{w}_k + \Lambda_k \underline{u}_k \tag{6}$$

where

$$\Phi_k = \Phi(t_{k+1}, t_k)$$

$$\Gamma_k \underline{w}_k = \int_{t_k}^{t_{K+1}} \Phi(t_{K+1}, \tau) G(\tau) \underline{w}(\tau) d\tau \tag{7}$$

$$\Lambda_k \underline{u}_k = \int_{t_k}^{t_{K+1}} \Phi(t_{K+1}, \tau) L(\tau) \underline{u}(\tau) d\tau \tag{8}$$

In general, Eqs. (7) and (8) provide unique definitions of only the products $\Gamma_k \underline{w}_k$ and $\Lambda_k \underline{u}_k$ and not the individual terms Γ_k, $\underline{w}_k$, Λ_k, $\underline{u}_k$. If $\underline{w}(\tau)$ is a vector of random process, $\underline{x}_k$ and $\Gamma_k \underline{w}_k$ will be vectors of random process. Equation (6) is illustrated in Fig. 2.

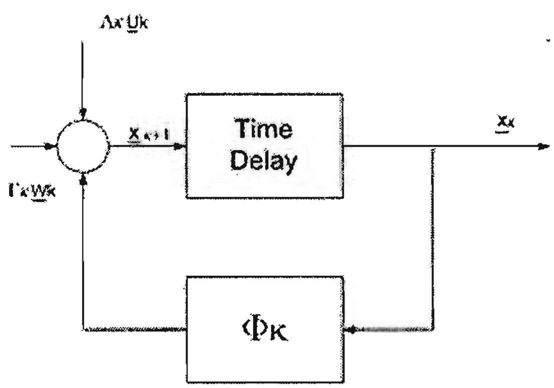


Fig.2 Illustration of Discrete Representation of Linear Dynamics Equation

3 SYSTEM OBSERVABILITY AND CONTROLLABILITY

Controllability deals with whether or not the state of a state-space equation can be controlled from the input, and observability deals with whether or not the initial state can be observed from the output.

3.1 OBSERVABILITY

Definition 1 *A system is observable at time $t_1 > t_0$, if it is possible to determine the state $\underline{x}(t_0)$ by observing $\underline{z}(t)$ in the interval (t_0, t_1). If all states $\underline{x}(t)$ corresponding to all $\underline{z}(t)$ are observable, the system is completely observable.*

Consider the discrete deterministic, constant n^{th}-order system

$$\underline{x}_{k+1} = \Phi_k \underline{x}_k$$

with n scalar noise-free measurements

$$z_k = H\underline{x}_k, \ \ k = 0, 1, 2, ..., n-1$$

so that H is a constant, n-dimensional row vector. We may write

$$\begin{aligned} z_0 &= H\underline{x}_0 \\ z_1 &= H\underline{x}_1 = H\Phi\underline{x}_0 \\ &\vdots \\ z_{n-1} &= H\underline{x}_{n-1} = H\Phi^{n-1}\underline{x}_0 \end{aligned}$$

Therefore,

$$\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} H \\ H\Phi \\ \vdots \\ H\Phi^{n-1} \end{bmatrix} \underline{x}_0 = \Xi^T \underline{x}_0$$

If $\underline{x}_0$ is to be determined uniquely, the matrix Ξ^T must be nonsingular. Thus, the observability condition is that the matrix

$$\Xi = \begin{bmatrix} H^T & \Phi^T H^T & \cdots & (\Phi^T)^{n-1} H^T \end{bmatrix}$$

be of rank n. The same condition can be extended to continuous system.

EXAMPLE

Consider the third-order system in Fig. 3 described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix}$$

If measurements can be made only at the output of the final integrator, then

$$z = x_3$$

so that

$$H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

we compute

$$H^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; F^T H^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; F^T F^T H^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

and form the matrix

$$\Xi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

A square $n \times n$ matrix has rank n if it has a nonzero determinant. The determinant of Ξ is zero so that the matrix has rank less than 3; thus the system is not observable. The physical interpretation of this result is that it is impossible to distinguish between the spectrally identical states x1 and x2 when the only available measurement is their sum.

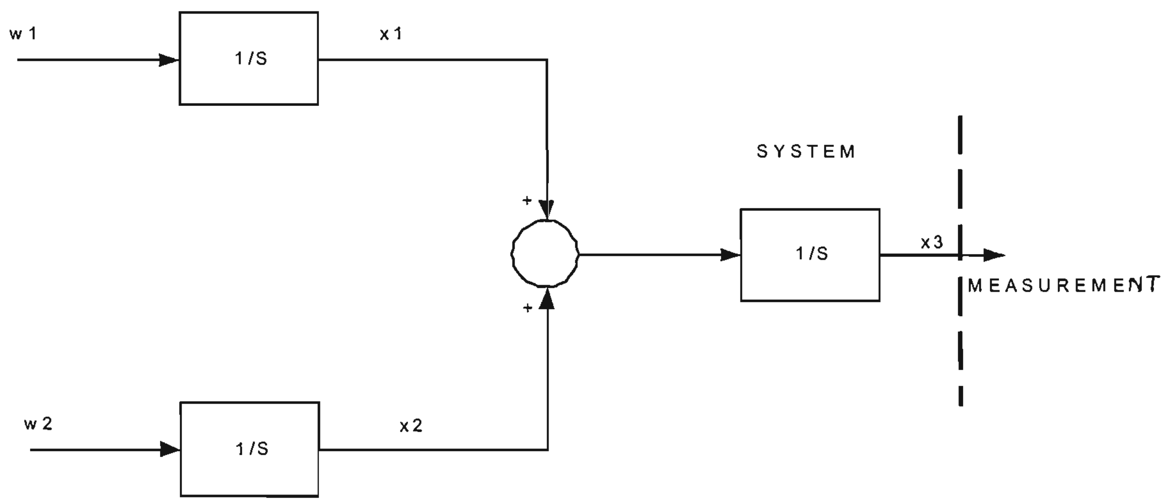


Fig. 3 Third-Order System with Output Observation Only

3.2 CONTROLLABILITY

Definition 2 *A system is controllable at time $t_1 > t_0$ if there exists a control $\underline{u}(t)$ such that any arbitrary state $\underline{x}(t_0) = \delta$ can be driven to another arbitrary state $\underline{x}(t_1) = \gamma$.*

For a constant discrete n^{th}-order system

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Lambda \underline{u}_k \tag{9}$$

The solution of (9) at $k = n$ can be derived as

$$\underline{x}_k = \Phi^k \underline{x}_0 + \sum_{m=0}^{k-1} \Phi^{k-1-m} \Lambda \underline{u}_m$$

which can be written as

$$\underline{x}_k - \Phi^k \underline{x}_0 = [\ \Lambda \quad \Phi\Lambda \quad \cdots \quad \Phi^{k-1}\Lambda\ ] \begin{bmatrix} \underline{u}_{k-1} \\ \underline{u}_{k-2} \\ \vdots \\ \underline{u}_0 \end{bmatrix}$$

for any $\underline{x}_k$ and $\underline{x}_0$, an input sequence exists if and only if the matrix

$$\Theta = [\ \Lambda \quad \Phi\Lambda \quad \cdots \quad \Phi^{k-1}\Lambda\ ]$$

is nonsingular.

EXAMPLE

Consider the network shown in Fig. 4. Its state variable x is the voltage across the capacitor. If $x(0) = 0$, then $x(t) = 0$ for all $t \geq 0$ no matter what input is applied. This is due to the symmetry of the network, and the input has no effect on the voltage across the capacitor. Thus the system is not controllable.

Next we consider the network shown in Fig. 5. It has two state variables $x1$ and $x2$ as shown. The input can transfer $x1$ or $x2$ to any value; but it cannot transfer $x1$ and $x2$ to any values. For example, if $x1(0)=x2(0) = 0$, then no matter what input is applied, $x1(t)$ always equals $x2(t)$ for all $t \geq 0$. Thus the equation that describes the network is not controllable.

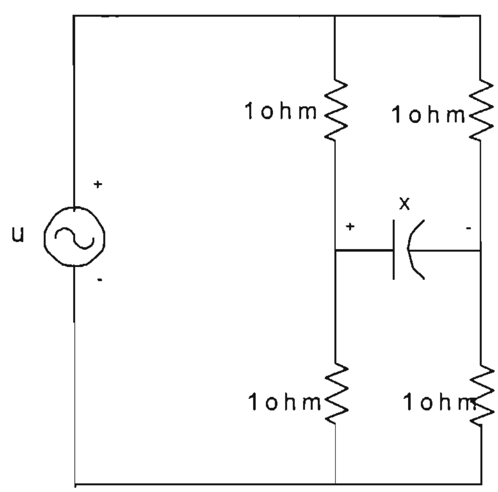


Fig. 4 Uncontrollable networks (a)

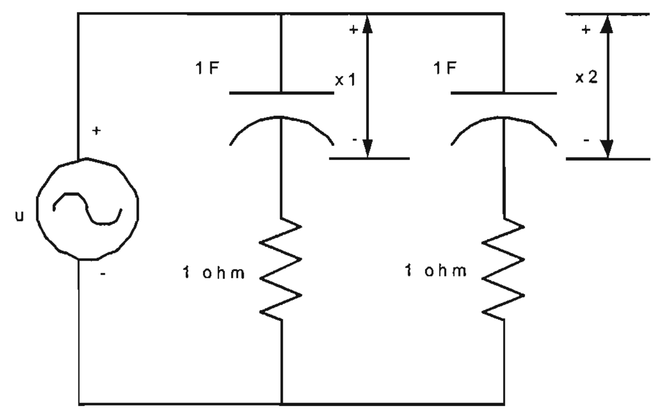


Fig. 5 Uncontrollable networks (b)

4 NONUNIQUENESS OF MODEL

For a continuous linear system, the general state-space model is

$$\begin{aligned}\dot{\underline{x}}(t) &= F(t)\underline{x}(t) + G(t)\underline{w}(t) \\ \underline{z}(t) &= H(t)\underline{x}(t) + \underline{v}(t)\end{aligned}$$

For a discrete linear system, the model is

$$\begin{aligned}\underline{x}_{k+1} &= \Phi_k \underline{x}_k + \Gamma_k \underline{w}_k \\ \underline{z}_k &= H_k \underline{x}_k + \underline{v}_k\end{aligned}$$

These models are not unique; given the pertinent system input and output quantities in the continuous case, there are many different sets of $F(t), G(t)$ and $H(t)$ which will yield the same overall input-output behavior.

EXAMPLE

The system with transfer function

$$G(s) = \frac{2S+1}{S^3 + 7S^2 + 14S + 8}$$

It can be written as

$$G(s) = \frac{-1/3}{S+1} + \frac{3/2}{S+2} + \frac{-7/6}{S+4}$$

The diagonal realization of the system is

$$
\begin{aligned}
\dot{\underline{x}}(t) &= \begin{bmatrix} -1 & & \\ & -2 & \\ & & -4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \underline{w}(t) \\
\underline{z}(t) &= \begin{bmatrix} -1/3 & 3/2 & -7/6 \end{bmatrix} \underline{x}(t)
\end{aligned}
$$

Another realization can be easily found, which is

$$
\begin{aligned}
\dot{\underline{x}}(t) &= \begin{bmatrix} 0 & 0 & -8 \\ 1 & 0 & -14 \\ 0 & 1 & -7 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \underline{w}(t) \\
\underline{z}(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \underline{x}(t)
\end{aligned}
$$

References:

1. Arthur Gelb(ed.). (1974). Applied Optimal Estimation. MIT Press.
2. Chi-Tsong Chen. (1998). Linear System Theory and Design. Oxford Univ. Press.
3. Wilson J. Rugh. (1995). Linear System Theory. Prentice Hall.