Static estimation problems.

1) Baye's Rule: $$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)}$$

2) Linear model of an observation process:

$$y = Hx + v \qquad y \in \mathbb{R}^k,\ x \in \mathbb{R}^n,\ v \in \mathbb{R}^k$$

$$v \sim N(0, V)$$
$$x \sim N(0, \Xi)$$

then $y \sim N(0, H\Xi H^T + V)$ (apriori)

? Given an observation of y, what is the distribution $p(x|y)$?

$p_{y|x}(y|x) =$ normal (Gaussian) $\sim N(0, V) = p_v(y - Hx)$

Answer:

$$p_{x|y}(x|y) = \frac{p(y|x)\,p(x)}{p(y)} = \frac{\left\{ \frac{1}{[(2\pi)^k |V|]^{1/2}} \exp\left[-\frac{1}{2}(y-Hx)^T V^{-1}(y-Hx)\right] \cdot \frac{1}{[(2\pi)^n |\Xi|]^{1/2}} \exp\left[-\frac{1}{2} x^T \Xi^{-1} x\right] \right\}}{\frac{1}{[(2\pi)^k |H\Xi H^T + V|]^{1/2}} \exp\left[-\frac{1}{2} y^T (H\Xi H^T + V)^{-1} y\right]}$$

$$\cancel{\begin{bmatrix} x \\ y \end{bmatrix}} \quad \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} m_x \\ m_y \end{bmatrix} \right)^T \Sigma_{xy}^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} m_x \\ m_y \end{bmatrix} \right)$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}^T \Sigma_{xy}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} - 2 \begin{bmatrix} x \\ y \end{bmatrix}^T \Sigma_{xy}^{-1} \begin{bmatrix} m_x \\ m_y \end{bmatrix} + \begin{bmatrix} m_x \\ m_y \end{bmatrix}^T \Sigma_{xy}^{-1} \begin{bmatrix} m_x \\ m_y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \underbrace{\begin{bmatrix} (H^T V^{-1} H + \Xi^{-1}) & -H^T V^{-1} \\ -V^{-1} H & V^{-1} \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Sigma_{xy}^{-1} \qquad \begin{bmatrix} (H^T V^{-1} H + \Xi^{-1}) & -H^T V^{-1} \\ -V^{-1} H & V^{-1} \end{bmatrix}$$

$$\cancel{(H^T V^{-1} H + \Xi^{-1})^{-1}}$$

$$\downarrow$$

$$m_{x|z} = K(y - Hx)$$

$$\min_K tr\left[E\left[(x - m_{x|z})(x - m_{x|z})^T\right]\right] = \min_K tr\left[E\left[xx^T - m_{x|z}x^T - x m_{x|z} + m_{x|z} m_{x|z}^T\right]\right]$$

$$= \min_K tr\left[E\left[(x - Ky + KHx)(x - Ky + KHx)\right]\right]$$

$$(x - K(y - Hx))(x - K(y - Hx))^T$$

$$E\left[xx^T - K(y - Hx)x^T - x(y - Hx)^T K^T + K(y - Hx)(y - Hx)^T K^T\right]$$

$$= \Gamma_x - K\Gamma_{vx} - \Gamma_{vx}K^T + K\Gamma_{vv}K^T$$

$$\frac{\partial}{\partial K} : \cancel{\Gamma_x} - 2\Gamma_{vx} + 2\Gamma_{vv}K = 0$$

$$K = \Gamma_{xv}\Gamma_{vv}^{-1}$$

$$\Gamma_{xv} = 0 \;!$$

$$\left[V^{-1} - V^{-1}H(H^TV^{-1}H + \Xi^{-1})^{-1}H^TV^{-1}\right]^{-1} = V\left[I - H(H^TV^{-1}H + \Xi^{-1})^{-1}H^TV^{-1}\right]^{-1}$$

$x, y \longrightarrow x, z$: independent.

$$z = y - \Gamma_{yx}\Gamma_{xx}^{-1}x \qquad xx^T\Gamma_{xx}^{-1}\Gamma_{xy}$$

$$E[xz^T] = E[xy^T - \cancel{x\Gamma_{yx}\Gamma_{xx}^{-1}\Gamma_{xy}x^T}]$$

$$= \Gamma_{xy} - \Gamma_{xy} = 0$$

$y = Hx + v$

$m_{y|x} = Hx + m_v$

$\Gamma_{y|x} = \Gamma_v$

$$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)} \leftarrow \begin{matrix} m_x \\ \Gamma_x \end{matrix}$$

$m_{x|y} = ?$

$\Gamma_{x|y} = ?$

$m_y = m_v + Hm_x$

$\Gamma_y = H\Gamma_xH^T + \Gamma_v$

$\cancel{m_{x|y} = [H^TV^{-1}H + \Gamma}$

$$m_{x|y} = [H^T\Gamma_x^{-1}H + \Gamma_v^{-1}]^{-1}H^T\Gamma_v^{-1}y$$

$$\Gamma_y^{-1} = H^T\Gamma_x^{-1}H + \Gamma_v^{-1}$$

$$m_{x|y} = \left[H^T V^{-1} H + \Xi^{-1}\right]^{-1} H^T V^{-1} y$$

$$\Sigma_{x|y}^{-1} = H^T V^{-1} H + \Xi^{-1}$$

$$\cancel{\frac{|V||\Xi|}{|H \Xi H^T + V|} = 1}$$

$$\cancel{p(x|y) \rightarrow} \qquad p(x,y) \longrightarrow p(x|y)$$

$$m_{x|y} = m_x + P_{xy} P_y^{-1} (y - m_y)$$

$$P_{x|y} = P_x - P_{xy} P_y^{-1} P_{yx}$$

$$p(x,y) = p(y|x)\, p(x)$$

$$= \frac{1}{\left[(2\pi)^k |V|\right]^{1/2}} \exp\left\{ -\frac{1}{2} (y - Hx)^T V^{-1} (y - Hx) \right\}$$

$$\cdot \frac{1}{\left[(2\pi)^n |\Xi|\right]^{1/2}} \exp\left\{ -\frac{1}{2} x^T \Xi^{-1} x \right\}$$

$$p(x,y) = \frac{1}{\left[(2\pi)^{n+k} |V||\Xi|\right]^{1/2}} \exp\left\{ -\frac{1}{2} \left\{ y^T V^{-1} y - 2 x^T H^T V^{-1} y + x^T H^T V^{-1} H x + x^T \Xi^{-1} x \right\} \right\}$$

$$= \quad '' \quad \exp\left\{ -\frac{1}{2} \left\{ x^T (H^T V^{-1} H + \Xi^{-1}) x - 2 x^T H^T V^{-1} y + y^T V^{-1} y \right\} \right\}$$

$$\cancel{\Sigma_{x|y}^{-1} = H^T V^{-1} H + \Xi^{-1}}$$

$$-\tfrac{1}{2}\,(\;) = x^T \Sigma_{x,y}^{-1} x - 2 x^T \Sigma_{x,y}^{-1} m_{x,y} + m_{xy}^T \Sigma_{x,y}^{-1} m_{xy}$$

$$= (x - m_{xy})$$

$$p_{x|y}(x|y) = \frac{1}{\left[(2\pi)^{n}|V|\,|\Xi|\,|H\Xi H^T+V|^{-1}\right]^{1/2}} \cancel{\exp\left[-\tfrac{1}{2}\right.}$$

$$\cdot \exp\left[-\frac{1}{2}\left[y^T V^{-1} y - y^T(H\Xi H^T+V)^{-1}y - 2x^T H^T V^{-1} y + x^T \Xi^{-1} x + x^T H^T V^{-1} H x\right]\right]$$

~~p(x,y~~

$$x^T\left(H^T V^{-1} H + \Xi^{-1}\right)x - 2x^T\left(H^T V^{-1} y\right) + b^T b$$

$$= (x - m_{x|y})^T \Sigma_{x|y}^{-1}(x - m_{x|y})$$

$$= x^T \Sigma_{x|y}^{-1} x - 2x^T \Sigma_{x|y}^{-1} m_{x|y} + \cancel{m_{x|y}^T m_{x|y}}\; m_{x|y}^T \Sigma_{x|y}^{-1} m_{x|y}$$

$$\Rightarrow \Sigma_{x|y}^{-1} = H^T V^{-1} H + \Xi^{-1}$$

$$\Sigma_{x|y}^{-1} m_{x|y} = H^T V^{-1} y$$

$$\Rightarrow m_{x|y} = \Sigma_{x|y} H^T V^{-1} y$$

$$= \left[H^T V^{-1} H + \Xi^{-1}\right]^{-1} H^T V^{-1} y$$

so $$b^T b = m_{x|y}^T \Sigma_{x|y}^{-1} m_{x|y} = y^T V^{-1} H \cancel{\Sigma_{x|y}}\, \cancel{\Sigma_{x|y}^{-1}}\, \Sigma_{x|y} H^T V^{-1} y$$

$$= y^T V^{-1} H \Sigma_{x|y} H^T V^{-1} y$$

$$= y^T V^{-1} H (H^T V^{-1} H + \Xi^{-1})^{-1} H^T V^{-1} y$$